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On the computation of analytical solutions of an unsteady magnetohydrodynamics flow of a third grade fluid with Hall effects

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ABSTRACT

In this article, a combination of Lie symmetry and homotopy analysis methods (HAM) are used to obtain solutions for the unsteady magnetohydrodynamics flow of an incompressible, electrically conducting third grade fluid, bounded by an infinite porous plate in the presence of Hall current. In particular, similarity reductions are performed on the governing equations in its complex scalar and corresponding vector system forms. Also, nontrivial conservation laws, using the *multiplier approach*, are constructed for the complex scalar equation. Furthermore, a comparison of the results with numerical results already existing in the literature is done. The analytical solutions are presented through graphs by choosing a range of the relevant physical parameters. The underlying calculations were obtained via a combination of software packages in Mathematica and Maple, in particular, for the Lie symmetry generators, Euler Lagrange operators and homotopy operators; the latter being towards the construction of the conserved flows.

1. Introduction

In various industrial applications, the notion of *magnetic field* in non-Newtonian fluid models has played a significant role. Some of these applications include magnetohydrodynamics (MHD) power generators, MHD flow meters, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays. Consequently, a significant amount of the literature is available. Some of the recent studies are given in [1–5]. However, the concept of considering Hall currents with magnetohydrodynamics flow especially with third grade fluids is relatively new. Several engineering applications in areas of Hall accelerator as well as in flight MHD have attracted the researchers and therefore some related studies [6–11] are available.

In this article, we will reconsider the problem of unsteady MHD flow of an incompressible electrically conducting third grade fluid, bounded by an infinite porous plate in the presence of Hall current, for analytical solutions. A combination of two different methods is used. First, a Lie symmetry analysis [12,13] is performed on the underlying model to reduce the system of partial differential equations (pdes) to a system of ordinary differential equations (odes). This is done through a number of reductions as the underlying Lie algebra of symmetries is multi-dimensional. Then, the HAM [14,15] approach is used on the reduced system to obtain a final solution of the model. For the computation of the Lie symmetry generators, Euler Lagrange operators and homotopy operators (used for the construction of the conserved flows), we adopt a range of computer packages like Mathematica and Maple. These extreme and tedious calculations are well nigh impossible by hand.

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The layout of the article is as follows. In Section 2, the statement of the flow problems is given. In Section 3, Lie symmetry analysis, reductions and the HAM for constructing solutions are done. Also, conservations laws for the scalar complex governing equations are determined. Results, discussions and comparisons are presented at the end.

2. Governing equations

Following the notations and preliminaries in [7], the equation governing the unsteady MHD flow of an incompressible electrically conducting third grade fluid in the presence of Hall current can be written as

$$\frac{\partial V}{\partial t} - W_0\frac{\partial V}{\partial y} = \nu\frac{\partial^2 V}{\partial y^2} + \frac{\alpha_1}{\rho}\left(\frac{\partial^3 V}{\partial y^2\partial t} - W_0\frac{\partial^3 V}{\partial y^3}\right) + \frac{6\beta_3}{\rho}\left(\frac{\partial V}{\partial y}\right)^2\frac{\partial^2 V}{\partial y^2} + \frac{\sigma M}{\rho\,(1-\mathrm{i}\psi)}\frac{\nu}{W_0^2}, \tag{1}$$

where $V = [u(y,t), -W_0, w(y,t)]$, u and w are the velocity components, $W_0 > 0$ corresponds to suction while $W_0 < 0$ represents blowing, α_i $(i = 1, 2)$, $\beta_i(1, 2, 3)$ are the material constants, B_0 is the uniform magnetic field, $M = B_0^2$, $\psi = \omega_e\tau_e$ is the Hall parameter with ω_e is the cyclotron frequency, τ_e is the electron collision time, μ is the dynamic viscosity, σ is the electrical conductivity, and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Further, the thermodynamics of the third grade fluid requires, the material constants and viscosity to satisfy the following conditions (see [16])

$$\begin{aligned}&\mu \ge 0, \qquad \alpha_1 \ge 0, \qquad |\alpha_1+\alpha_2| \le \sqrt{24\mu\beta_3},\\ &\beta_1 = \beta_2 = 0, \qquad \beta_3 \ge 0.\end{aligned}$$

Upon introducing the following dimensional variables $\alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, \alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, \alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, (u, w) = W_0(\bar{u}, \bar{w}), \varepsilon = \frac{6\beta_3}{\rho\nu^3}W_0^4$, Eq. (1), after dropping the bar, simplifies to

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial y^2} + \alpha\left(\frac{\partial^3 F}{\partial^2 y\partial t} - \frac{\partial^3 F}{\partial y^3}\right) + \varepsilon\left(\frac{\partial F}{\partial y}\right)^2\frac{\partial^2 F}{\partial y^2} - \kappa F, \tag{2}$$

where $\kappa = \frac{\sigma M}{\rho(1-\mathrm{i}\psi)}\frac{\nu}{W_0^2} = \frac{k}{(1-\mathrm{i}\psi)}$ and $F = u + \mathrm{i}w$. It should be noted that for third grade we assume $\beta_3 > 0$, otherwise the case $\beta_3 = 0$ implies $\alpha_1 + \alpha_2 = 0$, and hence (2) reduces to second grade fluid with Hall effect. The assumptions of $\psi = 0$ and $B_0 = 0$ will eliminate the effects of Hall term and uniform magnetic field. The separation of (2) into real and imaginary parts is given by

$$\begin{aligned}&u_t - u_y = u_{yy} + \alpha\left(u_{yyt} - u_{yyy}\right) + \varepsilon\left\{\left(u_y^2 - w_y^2\right)u_{yy} - 2u_y w_y w_{yy}\right\} - \frac{k}{1+\psi^2}\left(u - \psi w\right),\\ &w_t - w_y = w_{yy} + \alpha\left(w_{yyt} - w_{yyy}\right) + \varepsilon\left\{\left(u_y^2 - w_y^2\right)w_{yy} + 2u_y w_y u_{yy}\right\} - \frac{k}{1+\psi^2}\left(w + \psi u\right).\end{aligned} \tag{3}$$

3. Lie symmetries and HAM solutions

In this section, we find the Lie point symmetry generators and reductions of (3) and the complex scalar Eq. (2). In the latter case we do a complete solution of travelling form and, finally, a nontrivial conserved flow is constructed using the multiplier approach (see [17,18]).

3.1. Symmetries and reductions of (3)

It can be shown that the system (3), after detailed calculations, admits a four-dimensional Lie algebra of point symmetries with basis, in generator form, given by

$$\begin{aligned}&X_1 = \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \mathrm{e}^{\frac{-k}{1+\psi^2}t}\left[\cos\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial w} + \sin\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial u}\right],\\ &X_4 = \mathrm{e}^{\frac{-k}{1+\psi^2}t}\left[\sin\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial w} - \cos\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial u}\right].\end{aligned} \tag{4}$$

Any one of the generators or linear combinations in (4) would lead to a reduction of (3) to a system of odes; the standard one being the travelling wave type reduction given by $X = \frac{\partial}{\partial t} + K\frac{\partial}{\partial y}$. For illustrative purposes, we do a reduction of (3) using a combination of X_3 and X_2. The invariants of this combined operator are

$$x = y, \qquad w_* = w - \int \mathrm{e}^{\frac{-k}{1+\psi^2}t}\cos\left(\frac{k\psi}{1+\psi^2}t\right)\mathrm{d}t, \qquad u_* = u - \int \mathrm{e}^{\frac{-k}{1+\psi^2}t}\sin\left(\frac{k\psi}{1+\psi^2}t\right)\mathrm{d}t,$$

where $w_* = w_*(x)$ & $u_* = u_*(x)$. In these variables, (3) becomes the system of ode's

$$-u_*' = u_*'' - \alpha u_*''' + \varepsilon \left\{ \left(\left(u_*'\right)^2 - \left(w_*'\right)^2 \right) u_*'' - 2u_*' w_*' w_*'' \right\} - \frac{k}{1+\psi^2} \left(u_* - \psi w_*\right),$$
$$w_*' = -w_*'' + \alpha w_*''' + \varepsilon \left\{ \left(\left(u_*'\right)^2 - \left(w_*'\right)^2 \right) w_*'' + 2u_*' w_*' u_*'' \right\} + \frac{k}{1+\psi^2} \left(w_* + \psi u_*\right).$$

Similarly, the travelling wave reduction with $\xi = y - Kt$, $u_* = u$ and $w_* = w$ is

$$-\left(K+1\right) u_*' = u_*'' - \alpha \left(K+1\right) u_*''' + \varepsilon \left\{ \left(\left(u_*'\right)^2 - \left(w_*'\right)^2 \right) u_*'' - 2u_*' w_*' w_*'' \right\} - \frac{k}{1+\psi^2} \left(u_* - \psi w_*\right),$$
$$-\left(K+1\right) w_*' = -w_*'' - \alpha \left(K+1\right) w_*''' + \varepsilon \left\{ \left(\left(u_*'\right)^2 - \left(w_*'\right)^2 \right) w_*'' + 2u_*' w_*' u_*'' \right\} - \frac{k}{1+\psi^2} \left(w_* + \psi u_*\right).$$

The systems above can be further analysed using the Lie symmetries method or via some other more convenient approach such as the HAM.

3.2. Symmetries, reductions, solutions and conservation laws of (2)

We need not calculate the Lie point symmetries of (2) in the usual way (as above) since we get these by transforming the generators in (4) in terms of F and $\bar{F}$, viz.,

$$Y_1 = \frac{\partial}{\partial y}, \qquad Y_2 = \frac{\partial}{\partial t}, \qquad Y_3 = \mathrm{e}^{\frac{-k}{1+\psi^2} t} \left[-\mathrm{e}^{-\mathrm{i}\left(\frac{k\psi}{1+\psi^2} t\right)} \frac{\partial}{\partial F} + \mathrm{e}^{\mathrm{i}\left(\frac{k\psi}{1+\psi^2} t\right)} \frac{\partial}{\partial \bar{F}} \right],$$
$$Y_4 = -\mathrm{e}^{\frac{-k}{1+\psi^2} t} \left[\mathrm{e}^{-\mathrm{i}\left(\frac{k\psi}{1+\psi^2} t\right)} \frac{\partial}{\partial F} + \mathrm{e}^{\mathrm{i}\left(\frac{k\psi}{1+\psi^2} t\right)} \frac{\partial}{\partial \bar{F}} \right].$$

3.2.1. Travelling wave reduction and complete solution

Using $Y = \frac{\partial}{\partial t} + K \frac{\partial}{\partial y}$, which leads to the new invariant function $f = F(y, t)$, which will be the function of the single new independent variable $\xi = y - Kt$, where K represents the constant wave speed. With these change of variable along with $K = 1$ and $K = -1$, respectively, we have the following reduced equations for Eq. (2),

$$2\alpha f''' + [1 + \varepsilon f'(\xi)^2] f''(\xi) - 2f'(\xi) - \kappa f(\xi) = 0 \tag{5}$$

with boundary conditions $f(0) = c_0, f'(0) = c_1, f(+\infty) = 0$, and

$$[1 + \varepsilon f'(\xi)^2] f''(\xi) - \kappa f(\xi) = 0 \tag{6}$$

with boundary conditions $f(0) = c_0$ and $f(+\infty) = 0$.

By examining the (5) and (6), it is clear that Eq. (5) has both dispersion (represented through α_1) and diffusion terms (represented through β_3) while Eq. (4) represents the diffusion effects, only. From Eq. (3) one can expect to see how the nonlinear term affects the fluid flow motion.

Now we employ the HAM to get the solutions of nonlinear Eqs. (5) and (6). The velocity distribution $f(\xi)$ in a complex plane can be expressed by the set of base functions of the form

$$\left\{ \xi^k \exp(-n\xi) | k \geq 0, n \geq 0 \right\}, \tag{7}$$

in the form of the following homotopy-series

$$f(\xi) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k \xi^k \exp(-n\xi), \tag{8}$$

in which $a_{m,n}^k$ are the coefficients. Involving the so-called *Rule of solution expressions* for $f(\xi)$ and Eqs. (5) and (6), the initial guesses $f_0(\xi)$ and auxiliary linear operator $\mathbf{L}_1$ and $\mathbf{L}_2$ are

$$\begin{aligned} &f_0\left(\xi\right) = \left(2c_0 + c_1\right) \exp\left(-\xi\right) - \left(c_0 + c_1\right) \exp\left(-2\xi\right), \quad \text{for } K = 1, \\ &f_0\left(\xi\right) = c_0 \exp\left(-\xi\right), \quad \text{for } K = -1 \end{aligned} \tag{9}$$

$$\begin{aligned} &\mathbf{L}_1(f) = f''' + 3f'' + 2f', \quad \text{for } K = 1, \\ &\mathbf{L}_2(f) = f'' - f, \quad \text{for } K = -1, \end{aligned} \tag{10}$$

where

$$\mathbf{L}_1\left[C_1\exp(-\xi)+C_2\exp(-2\xi)+C_3\right]=0,\quad \text{for } K=1, \qquad \mathbf{L}_2\left[C_4\exp(-\xi)+C_5\exp(\xi)\right]=0,\quad \text{for } K=-1, \tag{11}$$

and $C_1 \ldots C_5$ are the constants. Eqs. (5) and (6) show that the nonlinear operators are

$$N_1\left[\hat{f}(\xi,p)\right]=2\alpha\frac{\partial^3\hat{f}(\xi,p)}{\partial\xi^3}+\frac{\partial^2\hat{f}(\xi,p)}{\partial\xi^2}-2\frac{\partial\hat{f}(\xi,p)}{\partial\xi}-\kappa\hat{f}(\xi,p)+\varepsilon\left(\frac{\partial\hat{f}(\xi,p)}{\partial\xi}\right)^2\frac{\partial^2\hat{f}(\xi,p)}{\partial\xi^2},\quad \text{for } K=1, \tag{12}$$

$$N_2\left[\hat{f}(\xi,p)\right]=\frac{\partial^2\hat{f}(\xi,p)}{\partial\xi^2}-\kappa\hat{f}(\xi,p)+\varepsilon\left(\frac{\partial\hat{f}(\xi,p)}{\partial\xi}\right)^2\frac{\partial^2\hat{f}(\xi,p)}{\partial\xi^2},\quad \text{for } K=-1. \tag{13}$$

Letting $\hbar_1$ and $\hbar_2$ as the non-zero convergence-control parameters, the *Zero*th order deformation problems are:

$$(1-p)\mathbf{L}_1[\hat{f}(\xi,p)-f_0(\xi)]=p\,\hbar_1\,N_1[\hat{f}(\xi,p)],\quad \text{for } K=1, \tag{14}$$

$$(1-p)\mathbf{L}_2[\hat{f}(\xi,p)-f_0(\xi)]=p\,\hbar_2\,N_1[\hat{f}(\xi,p)],\quad \text{for } K=-1, \tag{15}$$

$$\hat{f}(0,p)=c_0\quad \hat{f}'(0,p)=c_1,\quad \hat{f}(\infty,p)=0,\quad \text{for } K=1, \qquad \hat{f}(0,p)=c_0,\quad \hat{f}(\infty,p)=0,\quad \text{for } K=-1, \tag{16}$$

where $p\in[0,1]$ is a homotopy-parameter and for $p=0$ and $p=1$, we have

$$\hat{f}(\xi,0)=f_0(\xi),\quad \text{and}\quad \hat{f}(\xi,1)=f(\xi). \tag{17}$$

The initial guess $f_0(\xi)$ approach $f(\xi)$ when p varies from zero to unity. Hence by Taylor's series expansion we can write

$$\hat{f}(\xi,p)=f_0(\xi)+\sum_{m=1}^{\infty}f_m(\xi)\,p^m, \tag{18}$$

$$f_m(\xi)=\frac{1}{m!}\left.\frac{\partial^m\hat{f}(\xi,p)}{\partial p^m}\right|_{p=0} \tag{19}$$

and the convergence of the homotopy-series (18) depends upon $\hbar_1$ in case of $K=1$, and $\hbar_2$ in case of $K=-1$. The values of $\hbar_1$ and $\hbar_2$ are chosen in such a way that the series (18) is convergent at $p=1$, for both the cases of wave speed. Then by using (17) one obtains

$$f(\xi)=f_0(\xi)+\sum_{m=1}^{\infty}f_m(\xi). \tag{20}$$

Differentiating the zero-th order deformation m-times with respect to p and then dividing by $m!$ and finally setting $p=0$, we get the following mth-order deformation problems

$$\mathbf{L}_1\left[f_m(\xi)-\chi_m f_{m-1}(\xi)\right]=\hbar_1\,\mathbf{R}_m^1(\xi),\quad \text{for } K=1, \tag{21}$$

$$\mathbf{L}_2\left[f_m(\xi)-\chi_m f_{m-1}(\xi)\right]=\hbar_2\,\mathbf{R}_m^2(\xi)\quad \text{for } K=-1, \tag{22}$$

$$f_m(0)=f_m'(0)=f_m(\infty)=0,\quad \text{for } K=1, \qquad f_m(0)=f_m(\infty)=0,\quad \text{for } K=-1, \tag{23}$$

where

$$\mathbf{R}_m^1(\xi)=2\alpha f_{m-1}'''(\xi)+f_{m-1}''(\xi)-2f_{m-1}'(\xi)-kf_{m-1}(\xi)+\varepsilon\sum_{k=0}^{m-1}\sum_{l=0}^{k}f_{m-1-k}'f_{k-l}'f_l'',\quad \text{for } K=1 \tag{24}$$

$$\mathbf{R}_m^2(\xi)=f_{m-1}''(\xi)-kf_{m-1}(\xi)+\varepsilon\sum_{k=0}^{m-1}\sum_{l=0}^{k}f_{m-1-k}'f_{k-l}'f_l'',\quad \text{for } K=-1 \tag{25}$$

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{26}$$

The general solutions of Eqs. (21)–(26) are

$$f_m(\xi)=f_m^*(\xi)+C_1\exp(-\xi)+C_2\exp(-2\xi)+C_3,\quad \text{for } K=1 \qquad f_m(\xi)=f_m^*(\xi)+C_4\exp(-\xi)+C_5\exp(\xi),\quad \text{for } K=-1 \tag{27}$$

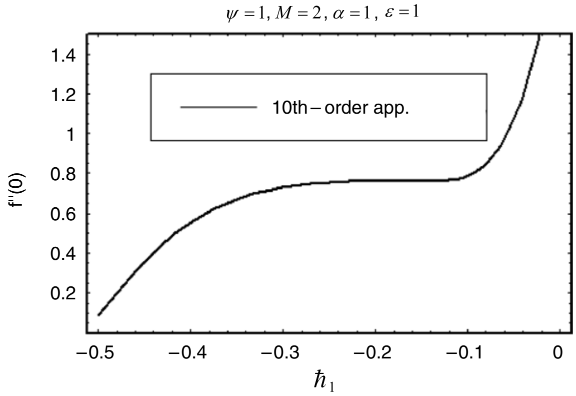

Fig. 1a. $\hbar$-curve for the function f for $K = 1$.

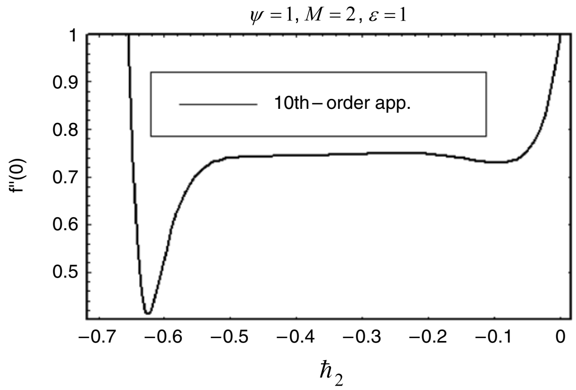

Fig. 1b. $\hbar$-curve for the function f for $K = -1$.

in which $f_m^*(\xi)$ is the particular solution and constants are determined by the boundary conditions (23)

$$C_3 = C_5 = 0, \qquad C_2 = f_m^*(0) + \left.\frac{\partial^2 f_m^*(\xi)}{\partial \xi^2}\right|_{\xi=0}, \qquad C_1 = -C_2 - f_m^*(0), \qquad C_4 = -f_m^*(0). \tag{28}$$

The Eqs. (21)–(26) can be solved by using the symbolic computation software *Mathematica* in the order $m = 1, 2, 3, \ldots$.

The homotopy-series solutions given by Eq. (20) involves the convergence-control parameters $\hbar_1$ and $\hbar_2$. The convergence of these homotopy-series solutions strongly depends upon the values of these parameters. One can choose a proper value of $\hbar_1$ and $\hbar_2$ to make the expressions (27) to be convergent by plotting the $\hbar$-curves. Figs. 1a and 1b show the $\hbar$-curves for f in the cases $K = 1$ and $K = -1$ respectively. It is observed that the HAM solutions given by Eq. (20) are convergent homotopy-series solutions in the whole region of ξ for $\hbar_1 = -0.25$ and $\hbar_2 = -0.35$.

3.2.2. Alternative reduction

Another nontrivial reduction of (2) is obtainable through a reduction using $Z = \frac{\partial}{\partial t} + Y_3$, having invariants

$$x, f_* = F + \frac{1+\psi^2}{k(-1+\mathrm{i}\psi)} \mathrm{e}^{\frac{k(-1+\mathrm{i}\psi)}{1+\psi^2} t}$$

where $f_* = f_*(x)$. A detailed calculation leads to the following reduction of (2) as a scalar ode

$$\alpha f_*''' - f_*'' - \varepsilon \left(f_*'\right)^2 f_*'' - f_*' + \kappa f_* = 0 \tag{29}$$

which admits only one Lie point symmetry $\frac{\partial}{\partial f_*}$, with invariants $g = f_*$ and $H = f_*'$. Under this change of variable (29) reduces to a second ode

$$\alpha \left(H^2 H'' + H \left(H'\right)^2\right) - HH' - \varepsilon H^3 H' - H - \kappa g = 0.$$

Again, this equation can be analysed using HAM as above or some other approach.

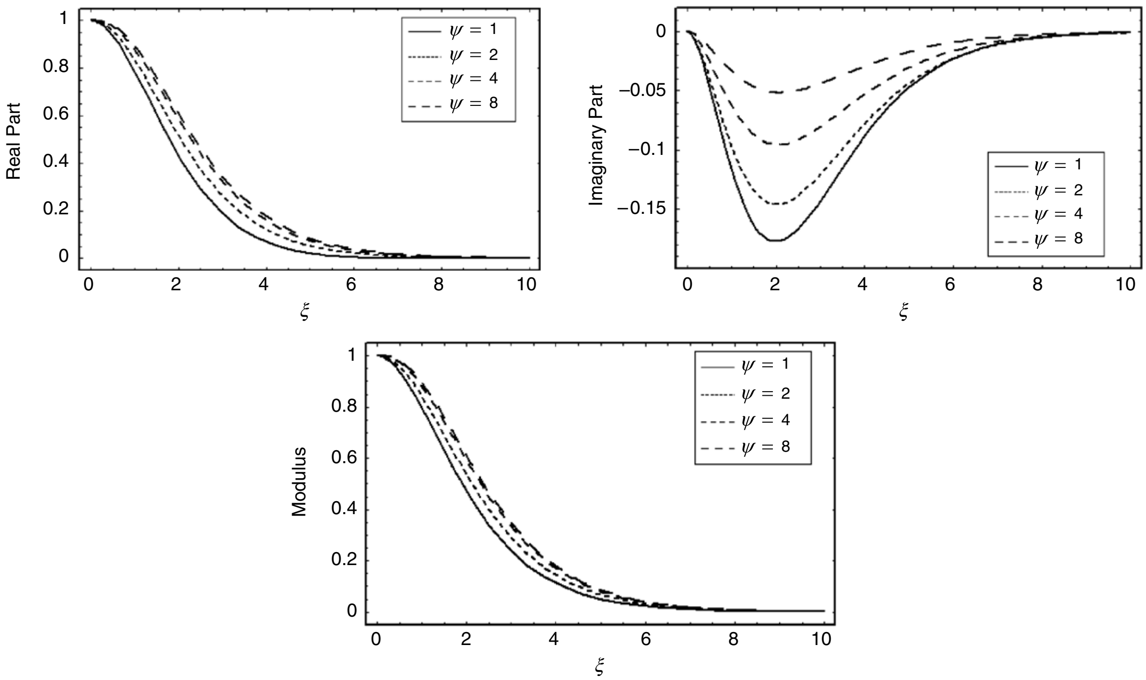

Fig. 2. Velocity profile $\alpha = 1$, $\varepsilon = 1$ and $M = 2$ with increasing ψ.

3.2.3. Conservation laws

We now determine the nontrivial conservation laws for (2). We achieve this by first multiplying (2) by a factor $Q = Q\left(x, t, F, F', F''\right)$ that satisfies

$$\frac{\delta}{\delta F}\left(Q(2)\right) = 0, \tag{30}$$

where $\frac{\delta}{\delta F}$ is the Euler–Lagrange operator. Eq. (30) is a consequence of the well known result that the Euler–Lagrange operator annihilates all total divergences—supposing that

$$\frac{\delta}{\delta F}\left(Q(2)\right) = D_t T^t + D_x T^x,$$

wherein (T^t, T^x) is the conserved vector of (2). The tedious set of calculations, not given here, associated with (30) leads to a single multiplier $Q = \mathrm{e}^{\kappa t}$. The components of the corresponding conserved vector is then calculated to be

$$T^t = \frac{1}{3}\mathrm{e}^{\kappa t}\left(3F - \alpha F_{xx}\right)$$

$$T^x = -\frac{1}{3}\mathrm{e}^{\kappa t}\left(3F + (3 - \alpha\kappa) F_x + \varepsilon F_x^3 + \alpha\left(2F_{xt} - 3F_{xx}\right)\right).$$

A vast amount of literature dealing with conservation laws is available. Relevant to the discussion above, the reader is referred to [19,20].

4. Concluding remarks

In this article, we have used the combination of Lie symmetries and HAM to reduce and solve the problem of unsteady MHD flow of an incompressible electrically conducting third grade fluid, bounded by an infinite porous plate under the influence of Hall effects. Inter alia, the corresponding governing equations have been reduced by travelling wave symmetries and, thereafter, HAM solutions have been obtained. The region of convergence for these solutions has also been done. The plots of the solutions have been presented for different values of physical parameters involved. In Figs. 2 and 3, we see the effect of the parameters representing second and third grade fluids, i.e., α_1 and β_3, which maybe regarded as dispersion and diffusion terms, respectively. Figs. 4 and 5 only reflect the effect of third grade fluid parameter β_3. We have compared the solutions with available perturbation solution given by [7] and the finite difference solution [11]. It is noted that the solutions obtained here are much better than the approximate perturbation solution and are in good agreement with the finite difference solution. However, it is noted that in [11], the values of the parameter chosen (e.g. $\varepsilon = 0, \ldots, 1000$) are

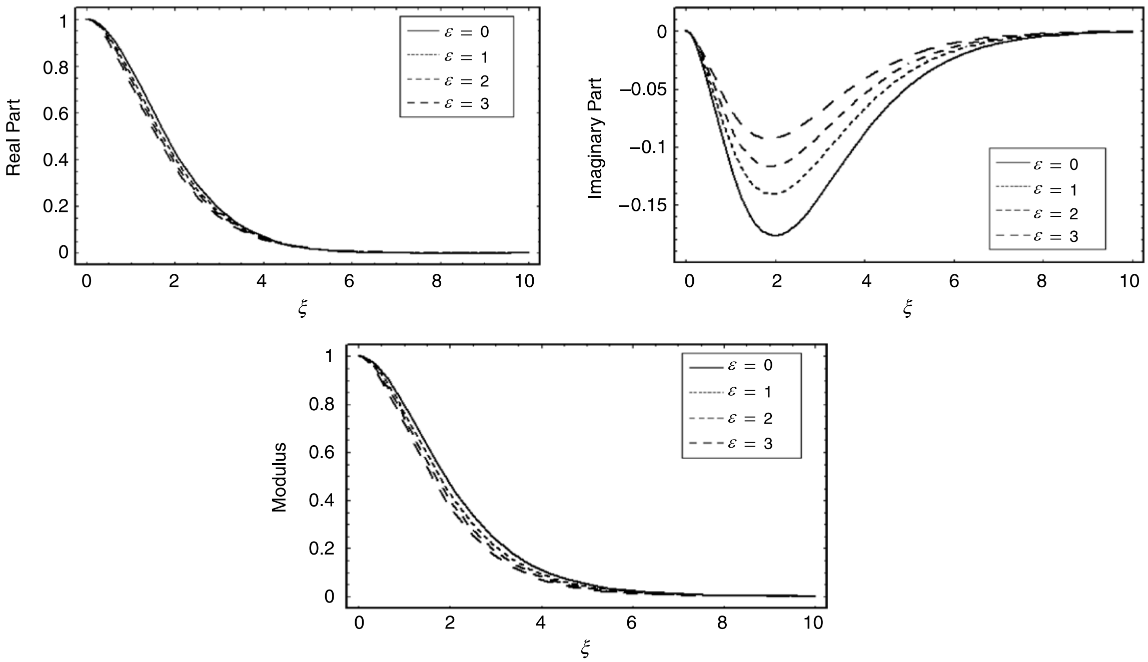

Fig. 3. Velocity profile $\alpha = 1$, $\psi = 1$ and $M = 2$ with increasing ε.

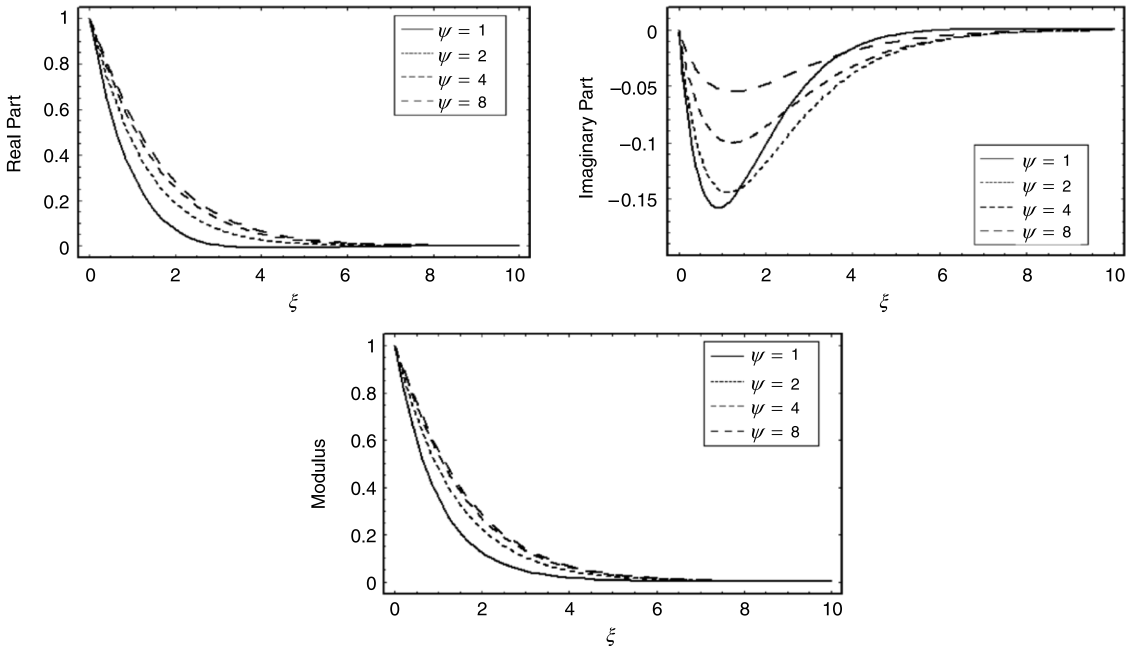

Fig. 4. Velocity profile $\varepsilon = 1$ and $M = 2$ with increasing ψ.

somewhat superficial and the domain of the ξ's are large so that some of the focus on the behavior of the graphs (results) are lost in the smaller part of the domain, whereas significance behaviors are visible in the presentation given here. We note that an Increase in the Hall parameter, ψ, results in the sharpening of the boundary layers, though consistent with Fig. 1(a) of [11] but very difficult to visualize, but this effect is obvious from Fig. 2 (real plot of the velocity profile), of our presentations. We can make similar other comments from a close study of the graphs presented here. In short, the results, via a combination of Lie symmetries and HAM allowed us to fine tune the profile of the results by using the practical values of the physical parameter and domain of the variables. Further, the procedure adopted here can be useful for handling a range of problems of a similar kind/nature.

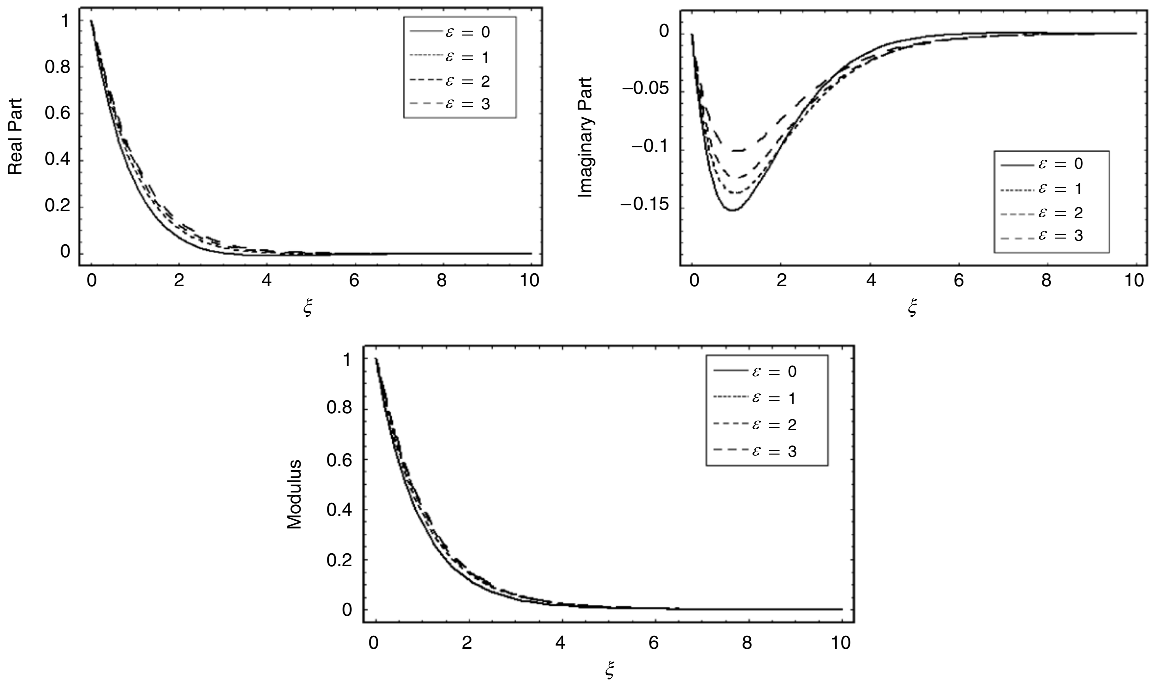

Fig. 5. Velocity profile $\psi = 1$ and $M = 2$ with increasing ε.

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